Un approccio moderno al teorema della mappa di Fueter-Sce-Qian

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A scientific day in honor of Michele Sce
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Fueter-Sce mapping theorem

Notations

- Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \ldots, e_n$ satisfying the relations

$$e_i e_j + e_j e_i = 0, \ \ i \neq j \qquad e_i^2 = -1.$$

- $\mathbb{R}_n$ is associative, non-commutative and for $n \geq 3$ it contains zero divisors
- An element in the Clifford algebra will be denoted by

$$\sum_A e_A x_A$$

where

$$A = \{i_1 \ldots i_r\} \in \mathcal{P}\{1, 2, \ldots, n\}, \ \ i_1 < \ldots < i_r$$

is a multi-index and $e_A = e_{i_1} e_{i_2} \ldots e_{i_r}$, $e_\emptyset = 1$.

Fueter-Sce mapping theorem

An element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element

$$x = x_0 + \underline{x} = x_0 + \sum_{j=1}^{n} x_j e_j \in \mathbb{R}_n$$

called, in short, paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as

$$|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2.$$

The real part x_0 of x is also denoted by $\mathrm{Re}(x)$; $\underline{x}$ is the 1-vector part of x; the conjugate of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{j=1}^{n} x_j e_j$.

Fueter-Sce mapping theorem

The sphere $\mathbb{S}^{n-1}$

$\mathbb{S}^{n-1} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \mid x_1^2 + \ldots + x_n^2 = 1\}$ $\underline{\omega} \in \mathbb{S}^{n-1}$, $\underline{\omega}^2 = -1$

The complex plane $\mathbb{C}_{\underline{\omega}}$

The vector space $\mathbb{R} + \underline{\omega}\mathbb{R}$ passing through 1 and $\underline{\omega} \in \mathbb{S}^{n-1}$ will be denoted by $\mathbb{C}_{\underline{\omega}}$, while an element belonging to $\mathbb{C}_{\underline{\omega}}$ will be denoted by $u + \underline{\omega}v$, for $u, v \in \mathbb{R}$. $\mathbb{C}_{\underline{\omega}}$ can be identified with a complex plane.

Fueter-Sce mapping theorem

Quaternions

When $n = 2$ we write $\mathbb{H}$ instead of $\mathbb{R}_2$, the basis is $1, i, j, k$ and $q \in \mathbb{H}$ is written as $q = x_0 + ix_1 + jx_2 + kx_3$, $x_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$.

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The sphere $\mathbb{S}$

$\mathbb{S} = \{\underline{x} = ix_1 + jx_2 + kx_3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ $\underline{\omega} \in \mathbb{S}$, $\underline{\omega}^2 = -1$

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Fueter-Sce mapping theorem

Let $x = x_0 + \underline{x}$; then $x = x_0 + \frac{\underline{x}}{|\underline{x}|}|\underline{x}|$ if $\underline{x} \neq \underline{0}$.
Given $x = u + \underline{\omega} v$ the set of all the elements of the form $x = u + Iv$ for $I \in \mathbb{S}^{n-1}$ is an $(n-1)$-sphere denoted by $[x]$.

A set Ω in $\mathbb{H}$ or $\mathbb{R}^{n+1}$ is said to be axially symmetric if $[x] \subset \Omega$ for any $x \in \Omega$.

Fueter-Sce mapping theorem

Cauchy-Fueter regular functions

Let $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ be a real differentiable function. We say that f is (left) Cauchy-Fueter regular if

$$\left(\frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \right) f = 0$$

in Ω.

- Gr. C. Moisil, Sur les quaternions monogènes, *Bull. Sci. Math. (Paris)*, **LV**, (1931).
- R. Fueter, *Analytische Funktionen einer Quaternionenvariablen*,Comment. Math. Helv. (1932).

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Monogenic functions

Let $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ be a real differentiable function. We say that f is (left) monogenic if

$$\left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + \ldots + e_n \frac{\partial}{\partial x_n}\right) f = 0$$

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- V. Iftimie, *Fonctions hypercomplexes*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, (1966).
- R. Delanghe, *On regular-analytic functions with values in a Clifford algebra*, Math. Ann. (1970).

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Fueter-Sce mapping theorem

Remark

- Cauchy-Fueter regular (resp. monogenic) functions are harmonic in 4 (resp. $n+1$) variables.
- Powers of the variable are not Cauchy-Fueter regular (resp. monogenic)
- Taylor and Laurent expansions can be obtained in terms of suitable homogeneous polynomials
- Results in classical complex analysis can be proved also in this framework

Fueter-Sce mapping theorem

Theorem (Fueter)

Let f be an holomorphic function in an open set of the upper half complex plane

$$f(u+\iota v)=\alpha(u,v)+\iota\beta(u,v)$$

$$q=x_0+ix_1+jx_2+kx_3:=x_0+\underline{q}$$

then we set

$$\alpha(x_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(x_0,|\underline{q}|)$$

and

$$\Delta_4\left(\alpha(x_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(x_0,|\underline{q}|)\right)$$

is Cauchy-Fueter regular.

Fueter-Sce mapping theorem

1. R. Fueter, *Die Funktionentheorie der Differentialgleichungen* $\Delta u = 0$ *und* $\Delta\Delta u = 0$ *mit vier reellen Variablen*, Comm.Math. Helv. (1934).

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Theorem (Sce)

Let f be an holomorphic function in an open set of the upper half complex plane

$$f(u+\iota v)=\alpha(u,v)+\iota\beta(u,v)$$

$$x=x_0+\underline{x}\in\mathbb{R}^{n+1}$$

then we set

$$\alpha(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}\beta(x_0,|\underline{x}|)$$

and for n odd

$$\Delta_{n+1}^{\frac{n-1}{2}}\left(\alpha(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}\beta(x_0,|\underline{x}|)\right)$$

is in the kernel of Dirac operator $\partial_x=\partial_{x_0}+\sum_i e_i\partial_{x_i}$, i.e. it is monogenic.

Fueter-Sce mapping theorem

1. M. Sce, *Osservazioni sulle serie di potenze nei moduli quadratici*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (1957).

Fueter-Sce mapping theorem

Generalizations

- Extension of Sce's results to the case n even:
  T. Qian, *Generalization of Fueter's result to* $\mathbb{R}^{n+1}$, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei(1997)
- Extension to the case of functions defined on open sets intersecting the real axis, symmetric and intrinsic functions, i.e. holomorphic functions such that $f(\bar{z}) = \overline{f(z)}$:
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Fueter-Sce mapping theorem

1. W. Sprössig, *On operators and elementary functions in Clifford analysis* Z. Anal. Anwendungen (1999).
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4. D. Peña-Peña, PhD dissertation, 2007-08.
5. N. Gürlebeck, *On Appell Sets and the Fueter-Sce Mapping*, Adv. Appl. Clifford Alg. (2009).
6. M. Fei, P. Cerejeiras, U. Kähler, *Fueter's theorem and its generalizations in Dunkl-Clifford analysis*, J. Phys. A, (2009).

Fueter-Sce mapping theorem

Remark

We have that

$$\mathbb{R}^{n+1} = \cup_{\underline{\omega}\in\mathbb{S}^{n-1}}\mathbb{C}_{\underline{\omega}}$$

and, similarly:

$$\mathbb{H} = \cup_{\underline{\omega}\in\mathbb{S}}\mathbb{C}_{\underline{\omega}}$$

Fueter-Sce mapping theorem

Slice regular functions

Let $U \subseteq \mathbb{H}$ be an open set and let $f : \ U \to \mathbb{H}$ be a real differentiable function.
Let $\underline{\omega} \in \mathbb{S}$ and let $f_{\underline{\omega}}$ be the restriction of f to the complex plane $\mathbb{C}_{\underline{\omega}}$.
We say that f is a (left) *slice regular function* if for every $\underline{\omega} \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + \underline{\omega}\frac{\partial}{\partial v}\right) f_{\underline{\omega}}(u + \underline{\omega} v) = 0.$$

We say that f is a *right slice monogenic function*, or right slice monogenic function, if for every $\underline{\omega} \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_{\underline{\omega}}(u + \underline{\omega} v) + \frac{\partial}{\partial v} f_{\underline{\omega}}(u + \underline{\omega} v)\underline{\omega}\right) = 0.$$

Fueter-Sce mapping theorem

Slice monogenic functions

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f: \ U \to \mathbb{R}_n$ be a real differentiable function.
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Fueter-Sce mapping theorem

Representation formula

Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Let f be a slice monogenic function on U. For any $x = x_0 + \underline{\omega}_x |\underline{x}| \in U$ the following formula holds:

$$f(x) = \frac{1}{2}\Big[1 - \underline{\omega}_x \underline{\omega}\Big] f(x_0 + \underline{\omega}|\underline{x}|) + \frac{1}{2}\Big[1 + \underline{\omega}_x \underline{\omega}\Big] f(x_0 - \underline{\omega}|\underline{x}|).$$

Fueter-Sce mapping theorem

Slice monogenic functions

Let $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ be a real differentiable function, where Ω is an axially symmetric open set. We say that f is (left) slice monogenic function if it is of the form

$$f(x + \underline{\omega} y) = \alpha(x, y) + \underline{\omega}\beta(x, y)$$

where $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$, and the pair α, β satisfies the Cauchy Riemann system, i.e. $\partial_x\alpha - \partial_y\beta = 0$, $\partial_y\alpha + \partial_x\beta = 0$. in Ω.

Fueter-Sce mapping theorem

Slice regular functions

Let $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ be a real differentiable function, where Ω is an axially symmetric open set. We say that f is (left) slice regular function if it is of the form

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Fueter-Sce mapping theorem

Theorem (Fueter)

Let

$$\alpha(x_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}\beta(x_0, |\underline{q}|)$$

be slice regular, then

$$\Delta_4 \left(\alpha(x_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}\beta(x_0, |\underline{q}|) \right)$$

is Cauchy-Fueter regular.

Fueter-Sce mapping theorem

Theorem (Sce)

Let

$$\alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|)$$

be slice monogenic, then for n odd

$$\Delta_{n+1}^{\frac{n-1}{2}}\left(\alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|)\right)$$

is in the kernel of Dirac operator $\partial_x = \partial_{x_0} + \sum_i e_i \partial_{x_i}$, i.e. it is monogenic.

The Cauchy formula with slice-monogenic kernel

Definition of noncommutative Cauchy kernel

We will call the expression

$$S^{-1}(s,x) = -(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}), \qquad (1)$$

defined for $x^2 - 2x\mathrm{Re}(s) + |s|^2 \neq 0$, noncommutative Cauchy kernel.

The Cauchy formula with slice-monogenic kernel

Proposition

$x^2 - 2x\mathrm{Re}(s) + |s|^2$ vanishes on the $(n-1)$-sphere

$$[s] = \{y = \mathrm{Re}(s) + I|\underline{s}|,\ \ I \in \mathbb{S}^{n-1}\}$$

Theorem

The function $S^{-1}(s,x)$ is left slice monogenic in the variable x and right slice monogenic in the variable s in its domain of definition.

The Cauchy formula with slice-monogenic kernel

Cauchy formula with slice monogenic kernel

Let $U \subset \mathbb{R}^{n+1}$ be a bounded axially symmetric s-domain such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of rectifiable Jordan curves for every $I \in \mathbb{S}^{n-1}$. Let f be a left slice monogenic function on $V \supset U$, $x \in U$ and set $ds_I = ds/I$, $ds = du + Idv$. Then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S^{-1}(s,x) ds_I f(s) \tag{2}$$

where

$$S^{-1}(s,x) = -(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s})$$

and the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}^{n-1}$.

The Cauchy formula with slice-monogenic kernel

Definition

Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$.

- We say that $S^{-1}(s,x)$ is written in the form I if

$$S^{-1}(s,x) := -(x^2 - 2x\mathrm{Re}(s) + |s|^2)^{-1}(x - \bar{s}).$$

- We say that $S^{-1}(s,x)$ is written in the form II if

$$S^{-1}(s,x) := (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}.$$

Theorem (Explicit computation of $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$)

Let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S^{-1}(s,x) = (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}$ be the slice-monogenic Cauchy kernel and let $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator in the variable x. Then, for $h \geq 1$, we have:

$$\Delta^h S^{-1}(s,x) = C_{n,h}(s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-(h+1)}.$$

where

$$C_{n,h} := (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n-(2\ell-1)).$$

The Fueter-Sce mapping theorem in integral form

Theorem

Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function $\Delta^h S^{-1}(s,x)$ is a right slice monogenic function in the variable s, for any h.

Theorem

Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Then the function $\Delta^{\frac{n-1}{2}} S^{-1}(s,x)$ is a monogenic function in the variable x.

The Fueter-Sce mapping theorem in integral form

Definition (The $\mathcal{F}_n$-kernel)

Let n be an odd number. Let $x, s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the $\mathcal{F}_n$-kernel as

$$\mathcal{F}_n(s,x) := \Delta^{\frac{n-1}{2}} S^{-1}(s,x) = \gamma_n(s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n-1)! \left(\frac{n-1}{2}\right)!.$$

The Fueter-Sce mapping theorem in integral form

Theorem (The Fueter mapping theorem in integral form)

Let n be an odd number. Let $W \subset \mathbb{R}^{n+1}$ be an axially symmetric open set and let $f \in \mathcal{SM}(W)$. Let U be a bounded axially symmetric open set such that $\overline{U} \subset W$. Suppose that the boundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}^{n-1}$. Then, if $x \in U$, the function $\tilde{f}(x)$ given by

$$\tilde{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$$

is monogenic and it admits the integral representation

$$\tilde{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n(s,x) ds_I f(s), \quad ds_I = ds/I, \tag{3}$$

where the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}^{n-1}$.

The inverse Fueter-Sce mapping theorem

Question

The Fueter mapping theorem implies the existence of a map:
$\mathcal{SM}(U) \to \mathcal{M}(U), \qquad f \mapsto \tilde{f}$. What is its image?

Definition (Axially monogenic function)

Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$, and let $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in U$. We say that $\tilde{f}$ is an axially monogenic function if there exist two functions $A = A(x_0, r)$ and $B = B(x_0, r)$, independent of $\underline{\omega} \in \mathbb{S}^{n-1}$ and with values in $\mathbb{R}_n$, such that

$$\tilde{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r),$$

and $\tilde{f}$ is a monogenic function, that is it is in the kernel of the Dirac operator. We denote by $\mathcal{AM}(U)$ the set of left axially monogenic functions on the open set U.

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The inverse Fueter-Sce mapping theorem

Theorem

Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. Then the functions $A = A(x_0, r)$ and $B = B(x_0, r)$ satisfy the Vekua's system, i.e.

$$\begin{cases} \partial_{x_0} A(x_0, r) - \partial_r B(x_0, r) = \frac{n-1}{r} B(x_0, r), \\ \partial_{x_0} B(x_0, r) + \partial_r A(x_0, r) = 0. \end{cases}$$

The inverse Fueter-Sce mapping theorem

Definition (The functions $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$)

Let $\mathcal{G}(x-\underline{y})$ be the monogenic Cauchy kernel with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$

$$\mathcal{G}(x) = \frac{1}{A_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \qquad x \in \mathbb{R}^{n+1} \setminus \{0\},$$

where $A_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$, and $\underline{\omega} \in \mathbb{S}^{n-1}$. We define

$$\mathcal{N}_n^+(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, dS(\underline{\omega}), \quad \mathcal{N}_n^-(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}).$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

The inverse Fueter-Sce mapping theorem

Theorem (The restrictions of $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$ to $\underline{x}=0$)

Let n be an odd number. Let $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ be the functions defined above. Then their restrictions to $\underline{x}=0$ are given by

$$\mathcal{N}_n^+(x)|_{\underline{x}=0} = \mathcal{C}_n \ \frac{x_0}{(x_0^2+1)^{(n+1)/2}}, \qquad \mathcal{N}_n^-(x)|_{\underline{x}=0} = -\mathcal{C}_n \ \frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where

$$\mathcal{C}_n := \frac{1}{\sqrt{\pi}} \ \frac{\Gamma((n+1)/2)}{\Gamma(n/2)}. \tag{4}$$

The inverse Fueter-Sce mapping theorem

The structure of the Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$

Let n be an odd number and denote by $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ the Fueter primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$, respectively. Consider the functions:

$$\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)}\frac{x_0}{(x_0^2+1)^{(n+1)/2}},$$

$$\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n}\ D^{-(n-1)}\frac{1}{(x_0^2+1)^{(n+1)/2}},$$

where the symbol $D^{-(n-1)}$ stands for the $(n-1)$ integrations with respect to x_0. Then replacing x_0 by x in $\mathcal{W}_n^+(x_0)$ and in $\mathcal{W}_n^-(x_0)$ we get $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$, respectively.

The inverse Fueter-Sce mapping theorem

Theorem (The inverse Fueter mapping theorem)

Let $\tilde{f}(x) = A(x_0, \rho) + \underline{\omega} B(x_0, \rho)$ be an axially monogenic function defined on an axially symmetric open set $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let $V = \{x = u + \underline{\omega} v,\ (u, v) \in \mathcal{V},\ \underline{\omega} \in \mathbb{S}^{n-1}\} \subset U$. Moreover suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $\rho = \rho(s)$. Then the function

$$f(x) = \int_\Gamma \mathcal{W}_n^- \Big(\frac{1}{\rho}(x - y_0)\Big) \rho^{n-2} (dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho))$$

$$- \int_\Gamma \mathcal{W}_n^+ \Big(\frac{1}{\rho}(x - y_0)\Big) \rho^{n-2} (dy_0 B(y_0, \rho) - d\rho A(y_0, \rho)). \qquad (5)$$

is a Fueter's primitive of $\tilde{f}(x)$ on V, where $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ are Fueter primitives of $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$, respectively.

The inverse Fueter-Sce mapping theorem

Theorem (The inverse Fueter mapping theorem for the quaternionic case)

Let $\tilde{f}(q) = A(q_0, \rho) + \underline{\omega} B(q_0, \rho)$ *be an axially Fueter regular function defined on an axially symmetric domain* $U \subseteq \mathbb{H}$. *Let* Γ *be the boundary of an open bounded subset* $\mathcal{V}$ *of the half plane* $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ *and let* $V = \{x = u + \underline{\omega} v,\ (u, v) \in \mathcal{V},\ \underline{\omega} \in \mathbb{S}^2\} \subset U$. *Moreover suppose that* Γ *is a regular curve . Then the function*

$$f(q) = \int_\Gamma \mathcal{W}^-\left(\frac{1}{\rho}(q - y_0)\right) \rho\,(dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho))$$

$$- \int_\Gamma \mathcal{W}^+\left(\frac{1}{\rho}(q - y_0)\right) \rho\,(dy_0 B(y_0, \rho) - d\rho A(y_0, \rho)). \qquad (6)$$

is a Fueter primitive of $\tilde{f}(q)$ *on* V, *where* $\mathcal{W}^+(q) = \frac{1}{2\pi}\ \arctan q$ *and* $\mathcal{W}^- = -\frac{1}{2\pi}\ q \arctan q$.

The inverse Fueter-Sce mapping theorem

Definition (Fueter's Primitive)

Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain. Let $\tilde{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0,\rho) + \underline{\omega}B(x_0,\rho))\mathcal{P}_k(\underline{x})$ be an axially monogenic function of degree $k \in \mathbb{N}_0$. We say that a function $f(x)\mathcal{P}_k(\underline{x})$, $f \in \mathcal{N}(U)$ is a Fueter primitive of $\tilde{f}(x)\mathcal{P}_k(\underline{x})$ if

$$\Delta^{k+\frac{n-1}{2}}(f(x)\mathcal{P}_k(\underline{x})) = \tilde{f}(x)\mathcal{P}_k(\underline{x}) \quad on \quad U,$$

where Δ is the Laplace operator in dimension $n+1$.

The inverse Fueter-Sce mapping theorem

Definition (The functions $\mathcal{F}^{+}_{k,n}(x)$ and $\mathcal{F}^{-}_{k,n}(x)$)

Let $\mathcal{G}(x-\underline{y})$ be the monogenic Cauchy kernel with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ and for $\underline{y} = r\underline{\omega} \in \mathbb{R}^n$ we assume $r=1$ and $\underline{\omega} \in \mathbb{S}^{n-1}$. Let $\mathcal{P}_k(\underline{x})$ be the inner left spherical monogenic polynomials of degree $k \in \mathbb{N}_0$. We define

$$\mathcal{F}^{+}_{k,n}(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \mathcal{P}_k(\underline{\omega})\, dS(\underline{\omega}),$$

$$\mathcal{F}^{-}_{k,n}(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x-\underline{\omega})\, \underline{\omega}\, \mathcal{P}_k(\underline{\omega}) dS(\underline{\omega}),$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

The inverse Fueter-Sce mapping theorem

Theorem (Factorization property of $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$)

Let n be an odd number. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomials of degree $k \in \mathbb{N}_0$. Then there exists two functions $\mathcal{S}^+_{k,n}(x)$ and $\mathcal{S}^-_{k,n}(x)$ belonging to $\mathcal{N}(U)$, independent of $\mathcal{P}_k(\underline{x})$, such that
$\mathcal{F}^+_{k,n}(x) = \mathcal{S}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$,
$\mathcal{F}^-_{k,n}(x) = \mathcal{S}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ *and*

$$\lim_{\underline{x}\to\underline{0}} \mathcal{S}^+_{k,n}(x) = \mathcal{C}_{k,n} \frac{x_0}{(x_0^2+1)^{k+(n+1)/2}},$$

$$\lim_{\underline{x}\to\underline{0}} \mathcal{S}^-_{k,n}(x) = -\mathcal{C}_{k,n} \frac{1}{(x_0^2+1)^{k+(n+1)/2}},$$

where $\mathcal{C}_{k,n} := \frac{(-1)^k}{\sqrt{\pi}} \frac{\Gamma(k+\frac{n+1}{2})}{\Gamma(k+\frac{n}{2})}$.

The inverse Fueter-Sce mapping theorem

Definition

Let n be an odd number. Let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomials of degree $k \in \mathbb{N}_0$. We will denote by $\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$ and $\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ the Fueter primitives of $\mathcal{F}^+_{k,n}(x)$ and $\mathcal{F}^-_{k,n}(x)$, that is $\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})$ and $\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})$ satisfy

$$\Delta^{k+\frac{n-1}{2}}(\mathcal{W}^+_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{F}^+_{k,n}(x), \quad \Delta^{k+\frac{n-1}{2}}(\mathcal{W}^-_{k,n}(x)\mathcal{P}_k(\underline{x})) = \mathcal{F}^-_{k,n}(x).$$

The inverse Fueter-Sce mapping theorem

$$\mathcal{W}^{+}_{k,n}(x_0) = \frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\ D^{-(2k+n-1)}\frac{x_0}{(x_0^2+1)^{k+(n+1)/2}}$$

$$\mathcal{W}^{-}_{k,n}(x_0) := -\frac{\mathcal{C}_{k,n}}{\mathcal{H}_{k,n}}\ D^{-(2k+n-1)}\frac{1}{(x_0^2+1)^{k+(n+1)/2}}.$$

Replacing now x_0 by x in $\mathcal{W}^{\pm}_{k,n}(x_0)$ we get $\mathcal{W}^{\pm}_{k,n}(x)$ which are the required functions.

The inverse Fueter-Sce mapping theorem

Theorem (The inverse Fueter-Sce mapping theorem)

Let n be an odd number and let $\mathcal{P}_k(\underline{x})$ be an inner left spherical monogenic polynomials of degree $k \in \mathbb{N}_0$. Let

$$\tilde{f}(x)\mathcal{P}_k(\underline{x}) = (A(x_0,\rho) + \underline{\omega}B(x_0,\rho))\mathcal{P}_k(\underline{x})$$

be an axially monogenic function of degree k defined on an axially symmetric $U \subseteq \Omega_n \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let $V \subset U$ be the open set in $\mathbb{R}^{n+1}$ induced by $\mathcal{V}$. Moreover suppose that Γ is a regular curve and consider the manifold

$$\Sigma := \{y_0 + \underline{\omega}\rho \,|\, (y_0,\rho) \in \Gamma, \quad \underline{\omega} \in \mathbb{S}^{n-1}\}.$$

The inverse Fueter-Sce mapping theorem

Then the function

$$f(x)\mathcal{P}_k(\underline{x}) = \int_\Gamma \mathcal{W}^-_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}[dy_0\, A(y_0,\rho) - d\rho\, B(y_0,\rho)]$$

$$-\int_\Gamma \mathcal{W}^+_{k,n}\Big(\frac{x-y_0}{\rho}\Big)\mathcal{P}_k\Big(\frac{\underline{x}-y_0}{\rho}\Big)\rho^{2k+n-2}[dy_0 B(y_0,\rho) + d\rho A(y_0,\rho)].$$

is a Fueter's primitive of $\tilde{f}(x)\mathcal{P}_k(\underline{x})$ on V.

The inverse Fueter-Sce mapping theorem

Let us denote by $\mathcal{AM}_k(U)$ the set of axially monogenic functions of degree k on the axially symmetric open set U and let us introduce the set

$$\mathscr{N}_k(U) = \{\varphi_k = \sum_{j=1}^{m_k} f_j(x)\mathcal{P}_{k,j}(\underline{x}) \ \mid f_j \in \mathcal{N}(U)\}.$$

The inverse Fueter-Sce mapping theorem

Corollary

Let n be an odd number and let U be an axially symmetric open set in $\mathbb{R}^{n+1}$. There is a map of $\mathbb{R}_n$-modules

$$\mathcal{AM}_k(U) \to \mathcal{N}_k(U),$$

such that $(A_k + \underline{\omega} B_k)\mathcal{P}_k = \Delta^{k+\frac{n-1}{2}}((\alpha_k + \underline{\omega}\beta_k)\mathcal{P}_k)$. *Moreover, there is a map*

$$\mathcal{M}(U) \to \bigoplus_k \Delta^k \mathcal{N}_k(U),$$

such that, given $\tilde{f} = \sum_k \tilde{f}_k \in \mathcal{M}(U)$, $f_k \in \mathcal{AM}_k(U)$, *there are* $\varphi_k \in \mathcal{N}_k$ *such that*

$$\tilde{f} = \Delta^{\frac{n-1}{2}} \sum_k \Delta^k \varphi_k.$$

The inverse Fueter-Sce mapping theorem

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