Un'applicazione del teorema della mappa di Fueter-Sce-Qian ad un calcolo funzionale monogenico

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Let us begin by recalling the classical Fueter-Sce-Qian theorem.

- R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*, Comm. Math. Helv., **7** (1934), 307–330.
- M. Sce, *Osservazioni sulle serie di potenze nei moduli quadratici*, Atti Acc. Lincei Rend. Fisica , **23** (1957), 220–225.
- T. Qian, *Generalization of Fueter's result to $\mathbb{R}^{n+1}$*, Rend. Mat. Acc. Lincei, **8** (1997), 111–117.

Theorem (Fueter)

Let f be a holomorphic function in an open set of the upper half complex plane and let $f(x+iy) = u(x,y) + iv(x,y)$, $\quad x, y \in \mathbb{R}$, where u and v are real differentiable functions with values in $\mathbb{R}$. Let

$$q = x_0 + ix_1 + jx_2 + kx_3 := x_0 + \underline{q}, \quad x_0, ..., x_3 \in \mathbb{R}$$

be a quaternion. Then Fueter's theorem asserts that the function defined by

$$\breve{f} := \Delta\Big(u(x_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(x_0, |\underline{q}|)\Big),$$

where Δ is the Laplace operator, is such that

$$\frac{\partial}{\partial \overline{q}} \breve{f} = \Big(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_3} + k\frac{\partial}{\partial x_3}\Big)\breve{f} = 0.$$

Theorem (Sce)

Consider the Euclidean space $\mathbb{R}^{n+1}$ whose variable is identified with a paravector $x_0 + \underline{x}$ where $\underline{x}$ is a 1-vector in the Clifford algebra $\mathbb{R}_n$, i.e. $\underline{x} = e_1 x_1 + \ldots + e_n x_n$. The Sce theorem in this setting states that, taken a function f as above, the function

$$\breve{f} := \Delta^{\frac{n-1}{2}} \Big(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \Big)$$

is monogenic, i.e. it is in the kernel of the operator

$$\frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

The proof of this result is due to Sce for n odd and to Qian for the case n even.

Function theories and spectral theories

We illustrate how complex analysis and operator theory extend to the quaternionic setting starting from the Fueter-Sce mapping theorem.

$$Hol(U) \xrightarrow{F_1} N(U) \xrightarrow{F_2=\Delta \quad (or \;\; F_2=\Delta^{(n-1)/2})} AM(U)$$

In $Hol(U)$ we have

- Holomorphic functions, Cauchy–Riemann $\partial_z = \partial_x + i\partial_y$, Cauchy formula, holomorphic functional calculus, spectral theorem
- Factorization of the Laplacian with ∂_z, Harmonic analysis
- Slice hyperholomorphicity $\mathcal{G} = |\underline{x}|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$
- Cauchy- Fueter regularity (or Dirac) $\mathcal{D} = \partial_{x_0} + \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$

$$SH(U) \xrightarrow{F_2=\Delta} AM(U)$$

$SH(U)$	$AM(U)$
Slice Cauchy Formula ↓	*Monogenic Cauchy Formula* ↓
↓	
S − *spectrum*, *SPECTR.TH.* ↓	*Monogenic spectrum*
↓	↓
S − *Functional calculus* ↓	*Monogenic Functional Calculus*
↓	↓
H^∞ − *Functional calculus* ↓	H^∞ − *Monogenic Functional Calculus*

The problem 1: Quantum Mechanics

- The interest in spectral theory for quaternionic operators is motivated by the paper:
  - G. Birkhoff, J. von Neumann, *The logic of quantum mechanics*, Ann. of Math., **37** (1936), 823-843.

  who showed that Schrödinger equation can be written basically in the complex or in the quaternionic setting.

- The first question is: what is the notion of spectrum of a quaternionic operator?
- What is the analogue of the Riesz-Dunford functional calculus?
- What is the analogue of the Spectral Theorem in the quaternionic setting?
- What about the quaternionic evolution operator?

The problem 2: New classes of diffusion problems

- Fractional heat equation

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0$$

- New approach: replace

$$\nabla = i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}$$

by ∇^α but for more general operators

$$\tilde{\nabla}(t,x) = (ia(x)\partial_{x_1} + jb(x)\partial_{x_2} + kc(x)\partial_{x_3})$$

to get

$$\partial_t u(t,x) + \mathrm{div}(\tilde{\nabla}(t,x))^\alpha u(t,x) = 0$$

The spectrum and resolvent operator

Let B be a linear bounded operator on a Banach space, we define the spectrum of T as

$$\sigma(B) = \{\lambda \in \mathbb{C} \ : \ \lambda I - B \ \textit{it is not invertible}\}$$

and the resolvent set

$$\rho(B) = \mathbb{C} \setminus \sigma(B).$$

The resolvent operator is defined as

$$R_B(\lambda) := (\lambda \mathcal{I} - B)^{-1}$$

for $\lambda \in \rho(B)$.

The classical case

- The Riesz-Dunford functional calculus. Let B be a bounded linear operator on a Banach space, and f be a holomorphic function, we define

$$f(B) = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathcal{I} - B)^{-1} f(\lambda)\, d\lambda,$$

  where Γ is a rectifiable Jordan curve that surrounds the spectrum $\sigma(B)$ of B.

- The spectral theorem. Let B be a bounded normal operator on a Hilbert space, and f be a continuous function then

$$f(B) = \int_{\sigma(B)} f(\lambda)\, dE(\lambda),$$

  where $dE(\lambda)$ are the spectral measures.

The quaternions

- We denote by $\mathbb{H}$ the algebra of quaternions. An element s is denoted by $s = s_0 + s_1 e_1 + s_2 e_2 + s_3 e_3$, where e_1, e_2 and e_3 are the imaginary units of the quaternion s,
- its conjugate is $\bar{s} = s_0 - s_1 e_1 - s_2 e_2 - s_3 e_3$,
- $\mathrm{Re}(s) = s_0$ is the real part,
- the norm $|s|$ is such that $|s|^2 = s_0^2 + s_1^2 + s_2^2 + s_3^2$.

The left and the right spectrum

- We denote by $\mathcal{B}(V)$ the left Banach space of all bounded right linear operators acting on the two sided quaternionic Banach space V.
- The left spectrum $\sigma_L(T)$ of $T \in \mathcal{B}(V)$ is defined by

$$\sigma_L(T) = \{s \in \mathbb{H} \quad : \quad s\mathcal{I} - T \quad \text{is not invertible in } \mathcal{B}(V)\},$$

  where the notation $s\mathcal{I}$ in $\mathcal{B}(V)$ means that $(s\mathcal{I})(v) = sv$ that is we have

$$T(v) - sv = 0$$

- The right spectrum $\sigma_R(T)$ of T is associated with the right eigenvalue problem, i.e., the search for those $s \in \mathbb{H}$ such that there exists a nonzero vector v satisfying

$$T(v) - vs = 0.$$

Remarks on the left and the right spectrum

- It is important to note that if s is a right eigenvalue, then all quaternions belonging to the sphere $r^{-1}sr$, $r \in \mathbb{H} \setminus \{0\}$, are also eigenvalues.
- But observe that the operator $\mathcal{I}s - T$ associated to the right eigenvalue problem is not linear, so it is not clear what is the resolvent operator to be considered.
- The left resolvent operator $\mathcal{R}_L(s, T)$ is defined by

  $$\mathcal{R}_L(s, T) := (s\mathcal{I} - T)^{-1}, \quad s \notin \sigma_L(T),$$

  but it is not known what notion of hyperholomorphicity it satisfies.
- In Adler's book:
  S. Adler, *Quaternionic Quantum Field Theory*, Oxford University Press, 1995, it is used the right spectrum of quaternionic operators.

Remark

Because of this ambiguity in the definition of the quaternionic spectrum one may be tempted to consider Fueter regular functions and see if their Cauchy kernel suggests what kind of resolvent operator and spectrum one should consider.

Definition (Fueter regular functions)

Let U be an open set in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is left Cauchy-Fueter (for brevity just Fueter) regular if

$$\frac{\partial}{\partial x_0} f(q) + e_1 \frac{\partial}{\partial x_1} f(q) + e_2 \frac{\partial}{\partial x_2} f(q) + e_3 \frac{\partial}{\partial x_3} f(q) = 0, \quad q \in U.$$

It is right Fueter regular if

$$\frac{\partial}{\partial x_0} f(q) + \frac{\partial}{\partial x_1} f(q) e_1 + \frac{\partial}{\partial x_2} f(q) e_2 + \frac{\partial}{\partial x_3} f(q) e_3 = 0, \quad q \in U.$$

The Cauchy kernel $\mathcal{G}$ and Fueter Cauchy formula

The Cauchy kernel $\mathcal{G}$

$$\mathcal{G}(s,q) = \frac{\bar{s}-\bar{q}}{|s-q|^4} = \frac{(s-q)^{-1}}{|s-q|^2} = (s-q)^{-2}(\bar{s}-\bar{q})^{-1}$$

and is both left and right Fueter regular on $\mathbb{H}\setminus\{0\}$.
Let $f \in \mathcal{R}_r$

$$f(q) := \frac{1}{2\pi^2}\int_{\partial\Omega} f(s) Ds\,\mathcal{G}(s,q),$$

where Ω is an open set in $\mathbb{H}$ containing the singularities, and Ds is a suitable differential form.

Cauchy kernel $\mathcal{G}$ admits the series expansion

The Cauchy kernel $\mathcal{G}$ admits the series expansion, (for $|s| < |q|$) of the form

$$\mathcal{G}(s,q) = \sum P_\nu(q) A_\nu(s), \tag{1}$$

where $A_\nu(s)$ are suitable coefficients and $P_\nu(q)$ denote the homogeneous polynomials

$$P_\nu(q) = \frac{1}{k!} \sum_{\ell_1,\ldots,\ell_k} z_{\ell_1} \ldots z_{\ell_k},$$

where if we denote by where

$$z_j = x_j - x_0 e_j, \quad for \quad j = 1, 2, 3,$$

for

$$q = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3.$$

The Fueter resolvent operator

When we replaces q by operator T and we suppose that $T = T_0 + e_1 T_1 + e_2 T_2 + e_3 T_3$, where the bounded operators T_ℓ, $\ell = 0, 1, 2, 3$ commute among themselves, then sum of the series (1) converges, for $\|T\| < |q|$, to

$$\mathcal{G}(s, T) = (s\mathcal{I} - T)^{-2}(\overline{s}\mathcal{I} - \overline{T})^{-1},$$

where $\overline{T} = T_0 - e_1 T_1 - e_2 T_2 - e_3 T_3$ so

$$\mathcal{G}(s, T) = \mathcal{R}_L(s, T)^2 \mathcal{R}_L(\overline{s}, \overline{T})$$

when $s \notin \sigma_L(T)$ and $\overline{s} \notin \sigma_L(\overline{T})$ and $\mathcal{G}(s, T)$ is Cauchy-Fueter regular operator-valued.

Remark: In the case T_ℓ, $\ell = 0, 1, 2, 3$ do not commute among themselves the sum is not known.

References for the Monogenic functional calculus

- B. Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics, 1843, Springer-Verlag, Berlin, 2004.
- B. Jefferies, A. McIntosh, *The Weyl calculus and Clifford analysis*, Bull. Austral. Math. Soc., **57** (1998), 329–341.
- B. Jefferies, A. McIntosh, J. Picton-Warlow, *The monogenic functional calculus*, Studia Math., **136** (1999), 99–119.
- A. McIntosh, A. Pryde, *A functional calculus for several commuting operators*, Indiana U. Math. J., **36** (1987), 421–439.
- C. Li, A. McIntosh, T. Qian, *Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces*, Rev. Mat. Iberoamericana, **10** (1994), 665–721.
- T. Qian, *Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space*, Math. Ann., **310** (1998), 601–630.

Recall the notations

- The unit sphere of purely imaginary quaternions

$$\mathbb{S} = \{q = e_1x_1 + e_2x_2 + e_3x_3 \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}.$$

  For $j \in \mathbb{S}$ we have $j^2 = -1$, and for this reason the elements of $\mathbb{S}$ are also called imaginary units.

- Given a nonreal quaternion $p = x_0 + \mathrm{Im}(p) = x_0 + j|\mathrm{Im}(p)|$, $j = \mathrm{Im}(p)/|\mathrm{Im}(p)| \in \mathbb{S}$, we can associate to it the 2-dimensional sphere defined by

$$[p] = \{x_0 + i|\mathrm{Im}(p)| \ : \ i \in \mathbb{S}\}.$$

- For any fixed $j \in \mathbb{S}$, we set $\mathbb{C}_j = \{u + jv \ : \ u, v \in \mathbb{R}\}$.

Definition

Let $U \subseteq \mathbb{H}$ be an open set. We say that U is axially symmetric if, for all $u + jv \in U$, the whole 2-sphere $[u + jv]$ is contained in U.

Definition (Left slice hyperholomorphic functions)

Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} \subseteq \mathbb{R} \times \mathbb{R}$ be such that $q = u + jv \in U$ for all $(u, v) \in \mathcal{U}$. We say that a functions on U of the form

$$f(q) = \alpha(u, v) + j\beta(u, v)$$

is left slice hyperholomorphic if α, β are $\mathbb{H}$-valued differentiable functions such that

$$\alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v) \quad \text{for all} \ \ (u, v) \in \mathcal{U}$$

and if α and β satisfy the Cauchy-Riemann system

$$\partial_u \alpha - \partial_v \beta = 0, \qquad \partial_v \alpha + \partial_u \beta = 0.$$

Definition (Right slice hyperholomorphic functions)

When f is of the form

$$f(q) = \alpha(u, v) + \beta(u, v)j$$

with the above properties for α and β we say that f is a right slice hyperholomorphic functions on U.

- $\mathcal{SH}_L(U)$ is the set of left slice hyperholomorphic functions on U
- $\mathcal{SH}_R(U)$ is the set of right slice hyperholomorphic functions on U.
- $\mathcal{N}(U)$ is the set of slice hyperholomorphic functions on U such that $\alpha(u, v)$ and $\beta(u, v)$ are real-valued (intrinsic functions)

Observation

$$\sum_{n\geq 0} q^n s^{-1-n} = \frac{1}{s-q}, \quad \textit{for} \quad q,\ s \in \mathbb{C}_j, \quad j \in \mathbb{S}, \quad |q| < |s|.$$

The problem is to replace q by T in

$$\sum_{n\geq 0} T^n s^{-1-n}?$$

and sum the series.

Slice hyperholomorphic Cauchy kernel

In the case of slice hyperholomorphicity the Cauchy kernel is given by the sum of the series

$$\sum_{n\geq 0} q^n s^{-1-n} = -(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q-\bar{s}), \quad \textit{for} \quad |q| < |s|$$

and it does not depend on the commutativity of the components of q. This crucial fact leads to the natural definition of the S-spectrum

$$\sigma_S(T) = \{s \in \mathbb{H} \ \ : \ \ T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \ \ \text{is not invertible in } \mathcal{B}(\mathrm{V})\}.$$

Definition

The left slice hyperholomorphic Cauchy kernel is

$$S_L^{-1}(s,q) = -(q^2 - 2\,\mathrm{Re}(s)q + |s|^2)^{-1}(q - \bar{s}) \quad \text{for } q \notin [s]$$

and the right slice hyperholomorphic Cauchy kernel is

$$S_R^{-1}(s,q) = -(q - \bar{s})(q^2 - 2\,\mathrm{Re}(s)q + |s|^2)^{-1} \quad \text{for } q \notin [s].$$

Theorem (Cauchy formulas)

Let $U \subset \mathbb{H}$ be an axially symmetric domain such that its boundary $\partial(U \cap \mathbb{C}_j)$ in $\mathbb{C}_j$ consists of a finite number of continuously differentiable Jordan curves. Let $j \in \mathbb{S}$ and set $ds_j = -j\, ds$. If f is left slice hyperholomorphic on an open set that contains $\overline{U}$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s) \quad \text{for all } q \in U.$$

If f is right slice hyperholomorphic on an open set that contains $\overline{U}$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q) \quad \text{for all } q \in U.$$

The above integrals depend neither on the open set U nor on the complex plane $\mathbb{C}_j$ for $j \in \mathbb{S}$.

Theorem (First crucial result)

Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$. Then, for $\|T\| < |s|$:

(1) $(T - \bar{s}\mathcal{I})^{-1} s (T - \bar{s}\mathcal{I}) - T$ *is the inverse of* $\sum_{n\geq 0} T^n s^{-1-n}$ *and*

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \quad (2)$$

(2) $(T - \bar{s}\mathcal{I}) s (T - \bar{s}\mathcal{I})^{-1} - T$ *is the inverse of* $\sum_{n\geq 0} s^{-1-n} T^n$ *and*

$$\sum_{n\geq 0} s^{-1-n} T^n = -(T - \bar{s}\mathcal{I})(T^2 - 2Re(s)T + |s|^2\mathcal{I})^{-1}. \quad (3)$$

Remark: *We do not require that when $T = T_0 + e_1 T_1 + e_2 T_2 + e_3 T_3$, the bounded operators T_ℓ, $\ell = 0, 1, 2, 3$ commute among themselves.*

Definition of the S-spectrum

- The quaternionic version of the Riesz-Dunford functional calculus requires the notion of S-spectrum
- Let $T \in \mathcal{B}(V)$. We define the S-spectrum $\sigma_S(T)$ of T as:

$$\sigma_S(T) = \{s \in \mathbb{H} \ \ : \ \ T^2 - 2s_0 T + |s|^2 \mathcal{I} \ \ \textit{is not invertible in } \mathcal{B}(V)\},$$

where $|s|^2 = s_0^2 + s_1^2 + s_2^2 + s_3^2$, and the S-resolvent set

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

Theorem (Properties of S-spectrum)

- *The S-spectrum is spherical symmetric and the S-eigenvalues are equal to the right eigenvalues.*
- *Let $T \in \mathcal{B}(V)$. Then the S-spectrum $\sigma_S(T)$ is a compact nonempty set.*

The S-resolvent operators

- Due to the noncommutativity of the quaternions, there are two resolvent operators associated with a quaternionic linear operator.
- The left S-resolvent operator

$$S_L^{-1}(s,T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \quad s \in \rho_S(T). \quad (4)$$

- The right S-resolvent operator

$$S_R^{-1}(s,T) := -(T - \bar{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}, \quad s \in \rho_S(T). \quad (5)$$

Theorem

Slice hyperholomorphicity of the S-resolvent operators

(i) *The left S-resolvent operator $S_L^{-1}(s,T)$ is a $\mathcal{B}(V)$-valued right-slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

(i) *The right S-resolvent operator $S_R^{-1}(s,T)$ is a $\mathcal{B}(V)$-valued left-slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

Formulations of the quaternionic functional calculus (CS)

- Let $U \subset \mathbb{H}$ be a suitable domain that contains the S-spectrum of T and set $ds_j = -j\, ds$. We define the quaternionic functional calculus for left slice hyperholomorphic functions $f : U \to \mathbb{H}$ as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \qquad (6)$$

- for right slice hyperholomorphic functions, we define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T). \qquad (7)$$

Theorem (Second crucial result)

These definitions are well posed since the integrals depend neither on the open set U nor on the complex plane $\mathbb{C}_j$.

Theorem (The S-resolvent equation (ACGS))

Let $T \in \mathcal{B}(V)$ and let s and $p \in \rho_S(T)$ such that $p \notin [s]$. Then we have

$$S_R^{-1}(s,T)S_L^{-1}(p,T) =$$
$$[(S_R^{-1}(s,T) - S_L^{-1}(p,T))p - \overline{s}(S_R^{-1}(s,T) - S_L^{-1}(p,T))](p^2 - 2s_0 p + |s|^2)^{-1}.$$

Theorem (The classical resolvent equation)

Let B be a complex operator on a Banach space. Then

$$(\lambda I - B)^{-1}(\mu I - B)^{-1} = \frac{(\lambda I - B)^{-1} - (\mu I - B)^{-1}}{\mu - \lambda}, \quad \lambda, \mu \in \rho(B), \quad (8)$$

$$S_R^{-1}(s,p) = -(p - \overline{s})(p^2 - 2s_0 p + |s|^2)^{-1} \quad \text{for } p \notin [s].$$

Some properties of the S-functional calculus

- The S-functional calculus agrees with that natural functional calculus for polynomials, thanks to the linearity and the fact that

$$T^m = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_i)} S_L^{-1}(s,T)\, ds_i\, s^m, \qquad m \in \mathbb{N} \cup \{0\}$$

  with obvious meaning of the symbols.
- In the case fg is slice slice hyperholomorphic we have the product rule $(fg)(T) = f(T)g(T)$ of the S-functional calculus.
- The spectral mapping theorem.
- The spectral radius theorem.
- Bounded perturbations of the functional calculus.
- Taylor formula for the S-functional calculus.

Main References

- F. C., I. Sabadini, *On some properties of the quaternionic functional calculus*, J. Geom. Anal., **19** (2009), 601-627.
- F. C., I. Sabadini, *On the formulations of the quaternionic functional calculus*, J Geom. Phys., **60** (2010), 1490–1508.
- D.Alpay, F. C., J. Gantner, and I. Sabadini, *A new resolvent equation for the S-functional calculus*, J. Geom. Anal., **25** (2015), 1939–1968.

The quaternionic functional calculus or S-functional calculus

- To replace the complex spectral theory with the quaternionic spectral theory we have to replace the classical spectrum with the S-spectrum.
- The S-functional calculus extends to the case of n-tuples of non commuting operators using slice monogenic functions.

The F-functional calculus for quaternionic operators

Theorem:
Let q, $s \in \mathbb{H}$ be such that $q \notin [s]$. Then the following identity holds:

$$-(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \bar{s}) = (s - \bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}.$$

Definition

Let q, $s \in \mathbb{H}$ be such that $q \notin [s]$.

- We say that $S_L^{-1}(s, q)$ is written in the form I if

$$S_L^{-1}(s, q) := -(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \bar{s}).$$

- We say that $S_L^{-1}(s, q)$ is written in the form II if

$$S_L^{-1}(s, q) := (s - \bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}.$$

The F-functional calculus for quaternionic operators

Definition (The $\mathcal{F}_L$-kernel)

Let q, $s \in \mathbb{H}$. We define, for $s \notin [q]$, the $\mathcal{F}$-kernel as

$$\mathcal{F}_L(s,q) := \Delta S_L^{-1}(s,q) = -4(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2},$$

Theorem (The Fueter mapping theorem in integral form)

$$\breve{f}(q) = \Delta f(q)$$

is Fueter regular and admits the integral representation

$$\breve{f}(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} \mathcal{F}_L(s,q)ds_j f(s), \quad ds_j = ds/j, \tag{9}$$

where the integral does not depend on U nor on the imaginary unit $j \in \mathbb{S}$.

Definition (The $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets)

Let $T \in \mathcal{BC}(V)$. We define the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ of T as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{H} \quad : \quad s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \quad \textit{is not invertible}\},$$

the $\mathcal{F}$-resolvent set $\rho_{\mathcal{F}}(T)$ is defined by

$$\rho_{\mathcal{F}}(T) = \mathbb{H} \setminus \sigma_{\mathcal{F}}(T)$$

where

$$T = T_0 + e_1 T_1 + e_2 T_2 + e_3 T_3, \qquad \overline{T} = T_0 - e_1 T_1 - e_2 T_2 - e_3 T_3$$

and T_ℓ, $\ell = 0, 1, 2, 3$ commute among themselves.

Theorem

$$\sigma_{\mathcal{F}}(T) = \sigma_S(T), \qquad T \in \mathcal{BC}(V).$$

The F-functional calculus for quaternionic operators

Definition

In the case of the F-functional calculus, let n be an odd number, we define the left F-resolvent operator as

$$F_L(s,T) := -4(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-2}, \quad s \notin \sigma_F(T) \quad (10)$$

and the right F-resolvent operator as

$$F_R(s,T) := -4(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-2}(s\mathcal{I} - \overline{T}), \quad s \notin \sigma_F(T). \quad (11)$$

The F-functional calculus for quaternionic operators

Definition (The quaternionic $\mathcal{F}$-functional calculus)

Let $T \in \mathcal{BC}(V)$. Let U be an open set, containing $\sigma_{\mathcal{F}}(T)$. Suppose that $f \in \mathcal{SH}^L_{\sigma_{\mathcal{F}}(T)}$ and let $\breve{f}(q) = \Delta f(q)$. We define the $\mathcal{F}$-functional calculus as

$$\breve{f}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} \mathcal{F}_L(s,T)\, ds_j\, f(s).$$

In the case of $f \in \mathcal{SH}^R_{\sigma_{\mathcal{F}}(T)}$

$$\breve{f}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, \mathcal{F}_R(s,T).$$

The F-Functional calculus

- F.C., I. Sabadini, F. Sommen, The Fueter mapping theorem in integral form and the $\mathcal{F}$-functional calculus, Mathematical Methods in the Applied Science, **33** (2010), 2050-2066.
- F.C., I. Sabadini, The F-functional calculus for unbounded operators, Journal of Geometry and Physics, **86** (2014), 392–407.
- F. C., J. Gantner, *Formulations of the F-functional calculus and some consequences*, Proceedings of the Royal Society of Edinburgh, 146 A, 1–37, 2016.

The McIntosh H^∞-functional calculus

- The Riesz-Dunford functional calculus. Let B be a bounded linear operator on a Banach space, and f be a holomorphic function, we define

$$f(B) = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathcal{I} - B)^{-1} f(\lambda)\, d\lambda,$$

where Γ is a rectifiable Jordan curve that surrounds the spectrum $\sigma(B)$ of B.

Operators of type ω

Let A be a linear operator on a complex Banach space X, with dense domain $\mathcal{D}(A)$ and dense range $\mathrm{Ran}(A)$. Let $\omega \in [0, \pi)$. We say that A is of type ω if

- its spectrum $\sigma(A)$ is contained in the sector

$$S_\omega = \{z \in \mathbb{C} \ : \ |\arg(z)| \leq \omega\} \cup \{0\}$$

- and if there exists a positive constant c_μ, for $\mu > \omega$, such that

$$\|(A - zI)^{-1}\| \leq \frac{c_\mu}{|z|},$$

  for all z such that $|\arg(z)| \geq \mu$.

For this class of operators it is possible to construct a functional calculus using bounded holomorphic functions g for which there exists two positive constants α and c such that

$$|g(z)| \leq \frac{c|z|^\alpha}{1+|z|^{2\alpha}} \quad \text{for all} \quad z \in S_\omega^0, \tag{12}$$

where S_ω^0 is the interior of S_ω.

The strategy is based on the Cauchy formula for holomorphic functions in which we replace the Cauchy kernel by the resolvent operator $R(\lambda, A)$.

$$g(A) = \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1} g(z) dz. \tag{13}$$

The integral turns out to be convergent for sectorial operators, if we assume that estimate (12) holds for the bounded holomorphic function g.

Now we extend the above calculus to functions

$$|f(z)| \leq c(|z|^k + |z|^{-k}), \quad \text{for} \quad c > 0, \; k > 0.$$

Definition of the H^∞ functional calculus

Using the functional calculus defined in (13) and the rational functional calculus

$$\varphi(A) = \Big(A(I + A^2)^{-1}\Big)^{k+1}, \quad k \in \mathbb{N}, \tag{14}$$

where

$$\varphi(z) = \Big(\frac{z}{1+z^2}\Big)^{k+1}, \quad k \in \mathbb{N},$$

we can define a more general functional calculus for sectorial operators given by

$$f(A) = (\varphi(A))^{-1}(f\varphi)(A) \tag{15}$$

where f is a holomorphic function on S^0_ω which satisfies bounds of the type

$$|f(z)| \le c(|z|^k + |z|^{-k}), \quad \text{for} \quad c > 0, \ k > 0.$$

- We could extend the H^∞-functional calculus to quaternionic operators (and also to n-tuples of noncommuting operators).
- We could use H^∞-functional calculus to give explicit formulas of the fractional powers T^α of a quaternionic operator T.

- D. Alpay, F.C., T. Qian, I. Sabadini, *The H^∞ functional calculus based on the S-spectrum for quaternionic operators and for n-tuples of noncommuting operators*, Journal of Functional Analysis, **271** (2016), 1544–1584.
- F. C. , J. Gantner: *An application of the S-functional calculus to fractional diffusion processes*, Milan Journal of Mathematics 2018.
- F. C., J. Gantner, *Fractional powers of quaternionic operators and Kato's formula using slice hyperholomorphicity*. Transactions American Mathematical Society (2018).
- M. Haase, *The functional calculus for sectorial operators*. Volume 169 Birkhäuser, Basel. 2006.
- A. McIntosh, *Operators which have an H^∞ functional calculus*. Proc. Centre Math. Anal. Austral. Nat. Univ., 14, Austral. Nat. Univ., Canberra, (1986).

Fractional Evolution

- u = temperature, $\mathbf{q}$ = heat flow, $k = 1$ thermal diffusivity

$$\mathbf{q} = -\nabla u \quad \text{(Fourier's law)}$$
$$\partial_t u + \operatorname{div} \mathbf{q} = 0 \quad \text{(Conservation of Energy)}$$

- Their combination yields the heat equation

$$\partial_t u - \Delta u = 0$$

- Alternative model: fractional heat equation

$$\partial_t u + (-\Delta)^\alpha u = 0$$

The main idea

- We identify

$$\mathbb{R}^3 \cong \{s \in \mathbb{H} : \mathrm{Re}(s) = 0\}$$

- We identify the gradient with the quaternionic nabla operator

$$\nabla = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3$$

- We replace the gradient in Fourier's law

$$u_t - \mathrm{div}(\nabla^\alpha u) = 0.$$

- Modifies flow, keeps conservation of energy, if this strategy works it is applicable to a large class of operators, for instance

$$\widehat{\nabla} = a(x)\partial_{x_1} e_1 + b(x)\partial_{x_2} e_2 + c(x)\partial_{x_3} e_3.$$

Technical problems and Workaround

Theorem

Consider ∇ on $L^2(\mathbb{R}^3, \mathbb{H})$. Then

$$\sigma_S(\nabla) = \mathbb{R}$$

- ∇^α cannot be defined because s^α is not defined on $(-\infty, 0)$
- workaround: define ∇^α only on the subspace associated to $[0, \infty)$ via

$$f_\alpha(\nabla)u = \frac{1}{2\pi} \int_{-j\mathbb{R}} S_L^{-1}(s, \nabla)\, ds_j\, s^\alpha \nabla u$$

for $u : \mathbb{R}^3 \to \mathbb{R}$ sufficiently regular; corresponds to Balakrishnan approach (deduced here by the quaternionic H^∞-functional calculus).

A surprising relation

- We have

$$S_L^{-1}(-tj,\nabla) = (-tj+\nabla)\underbrace{(-t^2+\Delta)^{-1}}_{=R_{-t^2}(-\Delta)}$$

- Some computations yield

$$\begin{aligned} f_\alpha(\nabla)u &= \frac{1}{2\pi}\int_{-j\mathbb{R}} S_L^{-1}(s,\nabla)\,ds_j\,s^\alpha \nabla u = \dots \\ &= \underbrace{\frac{1}{2}\nabla(-\Delta)^{\frac{\alpha}{2}-1}\nabla u}_{\mathrm{Scal} f_\alpha(\nabla)u} + \underbrace{\frac{1}{2}(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{\alpha}{2}-1}\nabla u}_{=\mathrm{Vec} f_\alpha(\nabla)u}. \end{aligned}$$

- We observe

$$\mathrm{div}\,\mathrm{Vec} f_\alpha(\nabla)u = -\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}+1}u$$

New fractional evolution equations

- The fractional heat equation for $\alpha \in (1/2, 1)$

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0$$

can hence be written as

$$\partial_t u(t,x) - 2\mathrm{div}\left(\mathrm{Vec} f_\beta(\nabla)u\right) = 0, \qquad \beta = 2\alpha - 1.$$

- For any suitable vector operator T, we hence propose

$$\partial_t u(t,x) - 2\mathrm{div}\left(\mathrm{Vec} f_\beta(T)u\right) = 0$$

as a fractional evolution equation.

A result with non constant coefficients

For

$$T = x_1\partial_{x_1} e_1 + x_2\partial_{x_2} e_2 + x_3\partial_{x_3} e_3$$

we have on $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu)$ with $\mathbb{R}^3_+ = \{x_1 e_1 + x_2 e_2 + x_3 e_3 : x_\ell > 0\}$ and $d\mu = \frac{1}{x_1 x_2 x_3} dx$ that

$$\mathrm{Vec} f_\beta(T)u(\xi) = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} -|y|^{2\alpha} e^{i\sum_{k=1}^3 \xi_k y_k} e^{-ix\cdot y} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix} dx\, dy.$$

The main applications of our theory

- The S-functional calculus: it is the quaternionic version of the Riesz-Dunford functional calculus for quaternionic operators and for n-tuples $(A_1, ..., A_n)$ of non commuting operators.
- The F-functional calculus based o the S-spectrum generates the McIntosh functional calculus for monogenic functions.
- The spectral theorem based on the S-spectrum has applications in quaternionic quantum mechanics.
- The quaternionic H^∞-functional calculus allows to define fractional powers of vector operators as (a,b,c real-valued functions)

$$\widehat{\nabla} = a(x)\partial_{x_1} e_1 + b(x)\partial_{x_2} e_2 + c(x)\partial_{x_3} e_3$$

so we have new fractional evolution problems.

(1) F. C., I. Sabadini, D. C. Struppa, *Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions*, Progress in Mathematics, Vol. 289, Birkhäuser, 2011, VI, 222 p.
(2) D. Alpay, F. C., I. Sabadini, *Slice Hyperholomorphic Schur analysis*, Series: Operator Theory: Advances and Applications, Vol. 256 (2017) Springer Basel, 365p.
(3) F. C., J. Gantner, D. P. Kimsey, *Spectral Theory on the S-spectrum for quaternionic operators*, Operator Theory: Advances and Applications, to appear in 2019.
(4) F. C., I. Sabadini, D. C. Struppa, *Entire slice regular functions*, SpringerBriefs in Mathematics. Springer, Cham, 2016. v+118 pp.
(5) S. G. Gal, I. Sabadini, *Quaternionic Approximation with application to slice regular functions*, Frontires in Mathematics, Birkäuser, to appear 2019.
(6) G. Gentili, C. Stoppato, D. C. Struppa, *Regular functions of a quaternionic variable*. Springer Monographs in Mathematics. Springer, Heidelberg, 2013, 185 p.